Algebra

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Algebra is a branch of mathematics that deals with abstract systems, known as algebraic structures, and the manipulation of expressions within those systems. It is a generalization of arithmetic that introduces variables and algebraic operations other than the standard arithmetic operations, such as addition and multiplication.

Elementary algebra is the main form of algebra taught in schools. It examines mathematical statements using variables for unspecified values and seeks to determine for which values the statements are true. To do so, it uses different methods of transforming equations to isolate variables. Linear algebra is a closely related field that investigates linear equations and combinations of them called systems of linear equations. It provides methods to find the values that solve all equations in the system at the same time, and to study the set of these solutions.

Abstract algebra studies algebraic structures, which consist of a set of mathematical objects together with one or several operations defined on that set. It is a generalization of elementary and linear algebra since it allows mathematical objects other than numbers and non-arithmetic operations. It distinguishes between different types of algebraic structures, such as groups, rings, and fields, based on the number of operations they use and the laws they follow, called axioms. Universal algebra and category theory provide general frameworks to investigate abstract patterns that characterize different classes of algebraic structures.

Algebraic methods were first studied in the ancient period to solve specific problems in fields like geometry. Subsequent mathematicians examined general techniques to solve equations independent of their specific applications. They described equations and their solutions using words and abbreviations until the 16th and 17th centuries when a rigorous symbolic formalism was developed. In the mid-19th century, the scope of algebra broadened beyond a theory of equations to cover diverse types of algebraic operations and structures. Algebra is relevant to many branches of mathematics, such as geometry, topology, number theory, and calculus, and other fields of inquiry, like logic and the empirical sciences.

*-algebra

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In mathematics, and more specifically in abstract algebra, a *-algebra (or involutive algebra; read as "star-algebra") is a mathematical structure consisting of two involutive rings R and A, where R is commutative and A has the structure of an associative algebra over R. Involutive algebras generalize the idea of a number system equipped with conjugation, for example the complex numbers and complex conjugation, matrices over the complex numbers and conjugate transpose, and linear operators over a Hilbert space and Hermitian adjoints.

However, it may happen that an algebra admits no involution.

Boolean algebra

mathematics and mathematical logic, Boolean algebra is a branch of algebra. It differs from elementary algebra in two ways. First, the values of the variables

In mathematics and mathematical logic, Boolean algebra is a branch of algebra. It differs from elementary algebra in two ways. First, the values of the variables are the truth values true and false, usually denoted by 1 and 0, whereas in elementary algebra the values of the variables are numbers. Second, Boolean algebra uses logical operators such as conjunction (and) denoted as ?, disjunction (or) denoted as ?, and negation (not) denoted as ¬. Elementary algebra, on the other hand, uses arithmetic operators such as addition, multiplication, subtraction, and division. Boolean algebra is therefore a formal way of describing logical operations in the same way that elementary algebra describes numerical operations.

Boolean algebra was introduced by George Boole in his first book The Mathematical Analysis of Logic (1847), and set forth more fully in his An Investigation of the Laws of Thought (1854). According to Huntington, the term Boolean algebra was first suggested by Henry M. Sheffer in 1913, although Charles Sanders Peirce gave the title "A Boolian [sic] Algebra with One Constant" to the first chapter of his "The Simplest Mathematics" in 1880. Boolean algebra has been fundamental in the development of digital electronics, and is provided for in all modern programming languages. It is also used in set theory and statistics.

C*-algebra

mathematics, specifically in functional analysis, a C?-algebra (pronounced "C-star") is a Banach algebra together with an involution satisfying the properties

In mathematics, specifically in functional analysis, a C?-algebra (pronounced "C-star") is a Banach algebra together with an involution satisfying the properties of the adjoint. A particular case is that of a complex algebra A of continuous linear operators on a complex Hilbert space with two additional properties:

A is a topologically closed set in the norm topology of operators.

A is closed under the operation of taking adjoints of operators.

Another important class of non-Hilbert C*-algebras includes the algebra

C

0

(

X

)

{\displaystyle C_{0}(X)}

of complex-valued continuous functions on X that vanish at infinity, where X is a locally compact Hausdorff space.

C*-algebras were first considered primarily for their use in quantum mechanics to model algebras of physical observables. This line of research began with Werner Heisenberg's matrix mechanics and in a more mathematically developed form with Pascual Jordan around 1933. Subsequently, John von Neumann attempted to establish a general framework for these algebras, which culminated in a series of papers on rings of operators. These papers considered a special class of C*-algebras that are now known as von Neumann algebras.

Around 1943, the work of Israel Gelfand and Mark Naimark yielded an abstract characterisation of C*-algebras making no reference to operators on a Hilbert space.

C*-algebras are now an important tool in the theory of unitary representations of locally compact groups, and are also used in algebraic formulations of quantum mechanics. Another active area of research is the program to obtain classification, or to determine the extent of which classification is possible, for separable simple nuclear C*-algebras.

Lie algebra

In mathematics, a Lie algebra (pronounced /li?/ LEE) is a vector space g {\displaystyle {\mathfrak {g}}} together with an operation called the Lie bracket

In mathematics, a Lie algebra (pronounced LEE) is a vector space

g

{\displaystyle {\mathfrak {g}}}

together with an operation called the Lie bracket, an alternating bilinear map

g

×

g

?

g

{\displaystyle {\mathfrak {g}}\times {\mathfrak {g}}\rightarrow {\mathfrak {g}}}

, that satisfies the Jacobi identity. In other words, a Lie algebra is an algebra over a field for which the multiplication operation (called the Lie bracket) is alternating and satisfies the Jacobi identity. The Lie bracket of two vectors

x

{\displaystyle x}

and

y

{\displaystyle y}

is denoted

[

x

,

y

]

{\displaystyle [x,y]}

. A Lie algebra is typically a non-associative algebra. However, every associative algebra gives rise to a Lie algebra, consisting of the same vector space with the commutator Lie bracket,

[

x

,

y

]

=

x

y

?

y

x

{\displaystyle [x,y]=xy-yx}

.

Lie algebras are closely related to Lie groups, which are groups that are also smooth manifolds: every Lie group gives rise to a Lie algebra, which is the tangent space at the identity. (In this case, the Lie bracket measures the failure of commutativity for the Lie group.) Conversely, to any finite-dimensional Lie algebra over the real or complex numbers, there is a corresponding connected Lie group, unique up to covering spaces (Lie's third theorem). This correspondence allows one to study the structure and classification of Lie groups in terms of Lie algebras, which are simpler objects of linear algebra.

In more detail: for any Lie group, the multiplication operation near the identity element 1 is commutative to first order. In other words, every Lie group G is (to first order) approximately a real vector space, namely the tangent space

g

{\displaystyle {\mathfrak {g}}}

to G at the identity. To second order, the group operation may be non-commutative, and the second-order terms describing the non-commutativity of G near the identity give

g

{\displaystyle {\mathfrak {g}}}

the structure of a Lie algebra. It is a remarkable fact that these second-order terms (the Lie algebra) completely determine the group structure of G near the identity. They even determine G globally, up to covering spaces.

In physics, Lie groups appear as symmetry groups of physical systems, and their Lie algebras (tangent vectors near the identity) may be thought of as infinitesimal symmetry motions. Thus Lie algebras and their representations are used extensively in physics, notably in quantum mechanics and particle physics.

An elementary example (not directly coming from an associative algebra) is the 3-dimensional space

g

=

R

3

{\displaystyle {\mathfrak {g}}=\mathbb {R} ^{3}}

with Lie bracket defined by the cross product

[

x

,

y

]

=

x

×

y

.

{\displaystyle [x,y]=x\times y.}

This is skew-symmetric since

x

×

y

=

?

y

×

x

{\displaystyle x\times y=-y\times x}

, and instead of associativity it satisfies the Jacobi identity:

x

×

(

y

×

z

)

+

y

×

(

z

×

x

)

+

z

×

(

x

×

y

)

=

0.

{\displaystyle x\times (y\times z)+\ y\times (z\times x)+\ z\times (x\times y)\ =\ 0.}

This is the Lie algebra of the Lie group of rotations of space, and each vector

v

?

R

3

{\displaystyle v\in \mathbb {R} ^{3}}

may be pictured as an infinitesimal rotation around the axis

v

{\displaystyle v}

, with angular speed equal to the magnitude

of

v

{\displaystyle v}

. The Lie bracket is a measure of the non-commutativity between two rotations. Since a rotation commutes with itself, one has the alternating property

[

x

,

x

]

=

x

×

x

=

0

{\displaystyle [x,x]=x\times x=0}

.

A Lie algebra often studied is not just the one associated with the original vector space, but rather the one associated with the space of linear maps from the original vector space. A basic example of this Lie algebra representation is the Lie algebra of matrices explained below where the attention is not on the cross product of the original vector field but on the commutator of the multiplication between matrices acting on that vector space, which defines a new Lie algebra of interest over the matrices vector space.

Algebra over a field

mathematics, an algebra over a field (often simply called an algebra) is a vector space equipped with a bilinear product. Thus, an algebra is an algebraic structure

In mathematics, an algebra over a field (often simply called an algebra) is a vector space equipped with a bilinear product. Thus, an algebra is an algebraic structure consisting of a set together with operations of multiplication and addition and scalar multiplication by elements of a field and satisfying the axioms implied by "vector space" and "bilinear".

The multiplication operation in an algebra may or may not be associative, leading to the notions of associative algebras where associativity of multiplication is assumed, and non-associative algebras, where associativity is not assumed (but not excluded, either). Given an integer n, the ring of real square matrices of order n is an example of an associative algebra over the field of real numbers under matrix addition and matrix multiplication since matrix multiplication is associative. Three-dimensional Euclidean space with multiplication given by the vector cross product is an example of a nonassociative algebra over the field of real numbers since the vector cross product is nonassociative, satisfying the Jacobi identity instead.

An algebra is unital or unitary if it has an identity element with respect to the multiplication. The ring of real square matrices of order n forms a unital algebra since the identity matrix of order n is the identity element with respect to matrix multiplication. It is an example of a unital associative algebra, a (unital) ring that is also a vector space.

Many authors use the term algebra to mean associative algebra, or unital associative algebra, or in some subjects such as algebraic geometry, unital associative commutative algebra.

Replacing the field of scalars by a commutative ring leads to the more general notion of an algebra over a ring. Algebras are not to be confused with vector spaces equipped with a bilinear form, like inner product spaces, as, for such a space, the result of a product is not in the space, but rather in the field of coefficients.

Associative algebra

In mathematics, an associative algebra A over a commutative ring (often a field) K is a ring A together with a ring homomorphism from K into the center

In mathematics, an associative algebra A over a commutative ring (often a field) K is a ring A together with a ring homomorphism from K into the center of A. This is thus an algebraic structure with an addition, a multiplication, and a scalar multiplication (the multiplication by the image of the ring homomorphism of an element of K). The addition and multiplication operations together give A the structure of a ring; the addition and scalar multiplication operations together give A the structure of a module or vector space over K. In this article we will also use the term K-algebra to mean an associative algebra over K. A standard first example of a K-algebra is a ring of square matrices over a commutative ring K, with the usual matrix multiplication.

A commutative algebra is an associative algebra for which the multiplication is commutative, or, equivalently, an associative algebra that is also a commutative ring.

In this article associative algebras are assumed to have a multiplicative identity, denoted 1; they are sometimes called unital associative algebras for clarification. In some areas of mathematics this assumption is not made, and we will call such structures non-unital associative algebras. We will also assume that all rings are unital, and all ring homomorphisms are unital.

Every ring is an associative algebra over its center and over the integers.

Abstract algebra

In mathematics, more specifically algebra, abstract algebra or modern algebra is the study of algebraic structures, which are sets with specific operations

In mathematics, more specifically algebra, abstract algebra or modern algebra is the study of algebraic structures, which are sets with specific operations acting on their elements. Algebraic structures include groups, rings, fields, modules, vector spaces, lattices, and algebras over a field. The term abstract algebra was coined in the early 20th century to distinguish it from older parts of algebra, and more specifically from elementary algebra, the use of variables to represent numbers in computation and reasoning. The abstract perspective on algebra has become so fundamental to advanced mathematics that it is simply called "algebra", while the term "abstract algebra" is seldom used except in pedagogy.

Algebraic structures, with their associated homomorphisms, form mathematical categories. Category theory gives a unified framework to study properties and constructions that are similar for various structures.

Universal algebra is a related subject that studies types of algebraic structures as single objects. For example, the structure of groups is a single object in universal algebra, which is called the variety of groups.

Exterior algebra

In mathematics, the exterior algebra or Grassmann algebra of a vector space V {\displaystyle V} is an associative algebra that contains V , {\displaystyle

In mathematics, the exterior algebra or Grassmann algebra of a vector space

V

{\displaystyle V}

is an associative algebra that contains

V

,

{\displaystyle V,}

which has a product, called exterior product or wedge product and denoted with

?

{\displaystyle \wedge }

, such that

v

?

v

=

0

{\displaystyle v\wedge v=0}

for every vector

v

{\displaystyle v}

in

V

.

{\displaystyle V.}

The exterior algebra is named after Hermann Grassmann, and the names of the product come from the "wedge" symbol

?

{\displaystyle \wedge }

and the fact that the product of two elements of

V

{\displaystyle V}

is "outside"

V

.

{\displaystyle V.}

The wedge product of

k

{\displaystyle k}

vectors

v

1

?

v

2

?

?

?

v

k

{\displaystyle v_{1}\wedge v_{2}\wedge \dots \wedge v_{k}}

is called a blade of degree

k

{\displaystyle k}

or

k

{\displaystyle k}

-blade. The wedge product was introduced originally as an algebraic construction used in geometry to study areas, volumes, and their higher-dimensional analogues: the magnitude of a 2-blade

v

?

w

{\displaystyle v\wedge w}

is the area of the parallelogram defined by

v

{\displaystyle v}

and

w

,

{\displaystyle w,}

and, more generally, the magnitude of a

k

{\displaystyle k}

-blade is the (hyper)volume of the parallelotope defined by the constituent vectors. The alternating property that

v

?

v

=

0

{\displaystyle v\wedge v=0}

implies a skew-symmetric property that

v

?

w

=

?

w

?

v

,

{\displaystyle v\wedge w=-w\wedge v,}

and more generally any blade flips sign whenever two of its constituent vectors are exchanged, corresponding to a parallelotope of opposite orientation.

The full exterior algebra contains objects that are not themselves blades, but linear combinations of blades; a sum of blades of homogeneous degree

k

{\displaystyle k}

is called a k-vector, while a more general sum of blades of arbitrary degree is called a multivector. The linear span of the

k

{\displaystyle k}

-blades is called the

k

{\displaystyle k}

-th exterior power of

V

.

{\displaystyle V.}

The exterior algebra is the direct sum of the

k

{\displaystyle k}

-th exterior powers of

V

,

{\displaystyle V,}

and this makes the exterior algebra a graded algebra.

The exterior algebra is universal in the sense that every equation that relates elements of

V

{\displaystyle V}

in the exterior algebra is also valid in every associative algebra that contains

V

{\displaystyle V}

and in which the square of every element of

V

{\displaystyle V}

is zero.

The definition of the exterior algebra can be extended for spaces built from vector spaces, such as vector fields and functions whose domain is a vector space. Moreover, the field of scalars may be any field. More generally, the exterior algebra can be defined for modules over a commutative ring. In particular, the algebra of differential forms in

k

{\displaystyle k}

variables is an exterior algebra over the ring of the smooth functions in

k

{\displaystyle k}

variables.

Algebra representation

In abstract algebra, a representation of an associative algebra is a module for that algebra. Here an associative algebra is a (not necessarily unital)

In abstract algebra, a representation of an associative algebra is a module for that algebra. Here an associative algebra is a (not necessarily unital) ring. If the algebra is not unital, it may be made so in a standard way (see the adjoint functors page); there is no essential difference between modules for the resulting unital ring, in which the identity acts by the identity mapping, and representations of the algebra.